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On exact solutions of the Schäfer–Wayne short pulse equation: WKI eigenvalue problem

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Abstract

We consider a new equation recently found by Schäfer and Wayne, hereafter named Schäfer–Wayne short pulse equation (SWSPE), describing the propagation of an ultrashort pulse in nonlinear media. Using some vanishing boundary conditions, we construct and discuss the N-soliton solutions to the previous equation by means of the Wadati–Konno–Ichikawa (WKI) method, which is arguably more direct than the map through the sine-Gordon equation investigated much earlier by Sakovich and Sakovich (2005 *J. Phys. Soc. Japan* **74** 239, 2006 *J. Phys. A: Math. Gen.* **39** L361). We particularly focus our attention on the two-soliton solution as an application. As a result, the collision process for two-soliton solutions with 'similar' amplitudes exhibits very different behaviour from the case when the amplitudes are 'dissimilar'.

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1. Introduction

In 1965, Zabusky and Kruskal discovered that the pulselike solitary wave solution to the Korteweg–de Vries (KdV) equation has a property which has been previously unknown, namely, that this solution interacts 'elastically' with another such solution. They termed these solutions solitons. Shortly after this discovery, Gardner *et al* [1, 2] proposed a new method of mathematical physics. Specifically, they have given a method of solution for the KdV equation by making use of the ideas of direct and inverse scattering. Lax [3] has considerably generalized these ideas, and Zakharov and Shabat [4] have shown that the method indeed works for another physically significant nonlinear evolution equation, namely, the nonlinear Schrödinger equation. Using these ideas, Ablowitz *et al* [5, 6] have developed a method to find a rather wide class of nonlinear evolution equations solvable by these techniques. They have termed the procedure the inverse scattering transform (IST). There have been numerous developments in 'solitons and IST' area, which have aroused considerable interest among mathematicians, physicists and engineers.

It is well known that solitons appear as a result of a balance between weak nonlinearity and dispersion. Soliton is defined as a nonlinear wave characterized by the following properties:

(i) a localized wave propagates without change of its properties (shape, velocity, etc);
(ii) localized waves are stable against mutual collisions and retain their identities.

Recently, in addition to the peakon solutions (a special type of weak solution of the (1+1)-dimensional Camassa–Holm (CH) equation [7]), some other types of weak solutions in nonlinear systems have attracted much attention. Among them, the so-called compacton solution is one of the most important excitations. Compacton solutions describe the typical (1+1)-dimensional soliton solutions with finite wavelength when the nonlinear dispersion effects are included in the model [8] and may have many interesting properties and possible physical applications [9]. For instance, the compacton equations may be used to study the motion of ion-acoustic waves and a flow of a two-layer liquid. On the other hand, Agüero and Paulin [10] have analysed the generalized ϕ^4 or double-well model with anharmonic interparticle interaction in the continuum limit by considering two types of boundary conditions: the trivial and the condensate types of boundary conditions at infinity. The most remarkable representatives of the structures they have found are the so-called drop compactons (solitons with compact support in the shape of hard spheres), cusps, loops, peak solitons (peakons) and defects.

Moreover, initial value problems for a number of nonlinear evolution equations, most of which are closely connected with physical problems, can be solved by means of the IST. Recently, Wadati *et al* [11] have found a new group of integrable nonlinear evolution equations while investigating a generalization of the IST. One member of this group is of the form

$$y_{xt} + \operatorname{sign}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)\left[\frac{y_{xx}}{\left(1+y_x^2\right)^{3/2}}\right]_{xx} = 0, \tag{1}$$

where x is the characteristic coordinate, t is the time along the coordinate, y is the spatial Cartesian and s is the arc length along the solution curve. The subscripts denote the partial differentiation. The IST scheme for equation (1) has been given by Konno *et al* [12–14], when only the consequences of a single loop soliton solution have been discussed. This IST scheme has been named Wadati–Konno–Ichikawa (WKI)-type eigenvalue problem.

The purpose of this paper is to discuss the WKI method for a recent equation known as the Schäfer–Wayne short pulse equation (SWSPE) [15] and derive the N-soliton solutions under the vanishing boundary conditions. Some interests are particularly paid for one- and two-soliton solutions as an application. It seems noteworthy to emphasize that Sakovich and Sakovich [16] may have found much earlier the N-soliton solutions to the SWSPE while constructing a map to the sine-Gordon equation. However, while investigating the two-soliton scattering behaviour, this map may not give more information concerning the ratio of the amplitudes of the soliton solutions.

Working with nonlinear evolution equations of the first type, as identified by WKI, Shimizu and Wadati [17] have derived the appropriate Gel'fand–Levitan equation and obtained the single soliton solution. In section 2, we shall modify their analysis to obtain the Gel'fand–Levitan equation for the system corresponding to the SWSPE. Then, in section 3, we shall obtain the N-soliton solution followed by two applications $N = 1$ and $N = 2$. As a result of this, we shall see that the collision process for two-soliton solutions with 'similar' amplitudes exhibits very different behaviour from the case when the amplitudes are 'dissimilar'. The collision process of a one soliton with positive amplitude and another of the same type with negative amplitude will also be discussed. Finally, in section 4, we will present a brief summary of the work.

2. The inverse scattering problem

The SWSPE is given by [15]

$$y_{xt} = y + (y^3)_{xx}/6, \tag{2}$$

where y is an observable and x and t are spatial and time-like coordinates, respectively. This equation provides a better approximation to the solution of Maxwell's equation. However, it seems to describe not only short pulses within a nonlinear medium but also multivalued solutions such as loop-like soliton.

Let us consider the following eigenvalue problem for the evolution equation (2):

$$v_{1x} + \imath\lambda v_1 = \lambda y_x v_2, \qquad v_{2x} - \imath\lambda v_2 = -\lambda y_x v_1, \tag{3}$$

where $\imath^2 = -1$, λ being the spectral parameter, and the time dependences of the eigenfunctions have the form

$$v_{1t} = A(\lambda, y, y_x)v_1 + B(\lambda, y, y_x)v_2, \qquad v_{2t} = C(\lambda, y, y_x)v_1 - A(\lambda, y, y_x)v_2. \tag{4}$$

Noting that $(v_{jx})_t = (v_{jt})_x$, $j = 1, 2$, and assuming that the eigenvalues λ are time-invariant, we readily find that $A(\lambda, y, y_x)$, $B(\lambda, y, y_x)$ and $C(\lambda, y, y_x)$ satisfy the following set of equations:

$$\begin{aligned} &A_x - \lambda(B + C)y_x = 0, \\ &\lambda y_{xt} - B_x - 2\imath\lambda B - 2\lambda A y_x = 0, \\ &\lambda y_{xt} + C_x - 2\imath\lambda C + 2\lambda A y_x = 0, \end{aligned} \tag{5}$$

which combined with equation (2) yields

$$A = \imath(1 - 2\lambda^2 y)/4\lambda, \qquad B = y(\lambda y y_x - \imath)/2, \qquad C = -y(\lambda y y_x + \imath)/2. \tag{6}$$

The boundary conditions are such that $y \to 0$ and $y_x \to 0$ as $|x| \to \infty$. The full inverse scattering method has been given in reference [2] by Konno *et al*, and will be used as the starting point of the resolution of the SWSPE, with the notation therein.

For real λ, the associated Jost functions may be defined as

$$\left.\begin{aligned} \phi &\to (1, 0)\exp(-\imath\lambda x), \\ \bar{\phi} &\to (0, -1)\exp(\imath\lambda x), \end{aligned}\right\} \qquad \text{for} \quad x \to -\infty \tag{7}$$

and

$$\left.\begin{aligned} \psi &\to (0, 1)\exp(\imath\lambda x), \\ \bar{\psi} &\to (1, 0)\exp(-\imath\lambda x), \end{aligned}\right\} \qquad \text{for} \quad x \to +\infty. \tag{8}$$

The scattering coefficients may also be defined as follows

$$\phi = a\bar{\psi} + b\psi, \qquad \bar{\phi} = -\bar{a}\psi + \bar{b}\bar{\psi}, \tag{9}$$

where

$$a\bar{a} + b\bar{b} = 1. \tag{10}$$

Then, for complex λ, it may be shown that

$$\bar{\phi}_1(\lambda) = \phi_2^\star(\lambda^\star), \qquad \bar{\phi}_2(\lambda) = -\phi_1^\star(\lambda^\star), \qquad \bar{\psi}_1(\lambda) = \psi_2^\star(\lambda^\star), \qquad \bar{\psi}_2(\lambda) = -\psi_1^\star(\lambda^\star), \tag{11}$$

where $(\star)$ denotes the complex conjugation. From equations (7)–(9), we get

$$\bar{a}(\lambda) = a^\star(\lambda^\star), \qquad \bar{b}(\lambda) = b^\star(\lambda^\star). \tag{12}$$

In order to examine the analytic properties of the Jost functions, we now follow Konno *et al* [18] and introduce

$$\phi_1 = \exp\left\{-\imath\lambda x + \int_{-\infty}^{x} \sigma(\lambda, s)\, \mathrm{d}s\right\}. \tag{13}$$

Substitution of equation (13) into (3) and (4) yields

$$\partial_t(y_x\Gamma) = \frac{1}{\lambda}\partial_x(A + B\Gamma), \qquad 2\imath\lambda(y_x\Gamma) = \lambda y_x^2 + \lambda(y_x\Gamma)^2 + y_x\partial_x(y_x\Gamma/y_x), \tag{14}$$

where $\Gamma = \phi_2/\phi_1$.

The first equation of the system (14) is in the form of conservation law. We expand $y_x\Gamma$ in the power series of $1/\imath\lambda$,

$$y_x\Gamma = \sum_{k=0}^{\infty} g_k(\imath\lambda)^{-k}. \tag{15}$$

Substituting equation (15) into the second equation of the system (14) and equating the terms of the same powers of $1/\lambda$, we obtain a recursion formula for the conserved densities g_k,

$$2\imath g_k = y_x^2\delta_{k,0}\imath^k + \sum_{n=0}^{k} g_n g_{k-n} + \imath y_x\partial_x(g_{k-1}/y_x). \tag{16}$$

The first three conserved densities are

$$\begin{aligned} g_0 &= \imath\left(1 - \epsilon\sqrt{1 + y_x^2}\right), \qquad \epsilon = \pm 1, \\ g_1 &= y_{xx}\left(1 - \epsilon\sqrt{1 + y_x^2}\right)/2y_x\sqrt{1 + y_x^2}, \\ g_2 &= -\frac{y_{xx}^2}{8\imath\left(1 + y_x^2\right)^{5/2}} - \partial_x\left\{\frac{y_x}{4\imath\left(1 + y_x^2\right)}\partial_x\left(\frac{1 - \sqrt{1 + y_x^2}}{y_x}\right)\right\}. \end{aligned} \tag{17}$$

With these $g_k (k = 0, 1, 2, \ldots)$, conservation laws for equation (2) are expressed by means of equation (14). By expanding σ in the power series of $1/\imath\lambda$ as

$$\sigma = \sum_{k=-1}^{\infty} \sigma_k(\imath\lambda)^{-k}, \tag{18}$$

we get

$$\sigma_k = g_{k+1}/\imath. \tag{19}$$

For example,

$$\sigma_{-1} = g_0/\imath, \qquad \sigma_0 = g_1/\imath. \tag{20}$$

The asymptotic behaviour of ϕ for large $|\lambda|$ may be written as

$$\phi = (1, \imath\sigma_{-1}/y_x)\exp\left\{-\imath\lambda x + \imath\lambda\int_{-\infty}^{x}\sigma_{-1}\,\mathrm{d}s + \int_{-\infty}^{x}\sigma_0\,\mathrm{d}s\right\} + o(1/\lambda). \tag{21}$$

The analytic property of the scattering cœ fficient a may be characterized by writing

$$a = \lim_{x\to\infty} \phi_1 \exp(\imath\lambda x) = \exp\left\{\imath\lambda \int_{-\infty}^{\infty} \sigma_{-1}\,\mathrm{d}s + \int_{-\infty}^{\infty} \sigma_0\,\mathrm{d}s\right\} + o(1/\lambda). \tag{22}$$

In the same way, we can obtain the asymptotic behaviour of $\bar\phi$, ψ and $\bar\psi$. We summarize these results as follows:

$$\begin{aligned}
\phi &= (1, \imath\sigma_{-1}/y_x)\exp(-\imath\lambda x + \imath\lambda\varepsilon_- + \mu_-) + o(1/\lambda),\\
\bar\phi &= (-\imath\sigma_{-1}/y_x, -1)\exp(\imath\lambda x - \imath\lambda\varepsilon_- + \mu_-) + o(1/\lambda),\\
\psi &= (\imath\sigma_{-1}/y_x, 1)\exp(\imath\lambda x + \imath\lambda\varepsilon_+ - \mu_+) + o(1/\lambda),\\
\bar\psi &= (1, \imath\sigma_{-1}/y_x)\exp(-\imath\lambda x - \imath\lambda\varepsilon_+ - \mu_+) + o(1/\lambda),\\
a &= \exp(\imath\lambda\varepsilon + \mu) + o(1/\lambda),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\varepsilon_- &= \int_{-\infty}^{x} \sigma_{-1}\,\mathrm{d}s, & \mu_- &= \int_{-\infty}^{x} \sigma_0\,\mathrm{d}s, & \varepsilon &= \int_{-\infty}^{\infty} \sigma_{-1}\,\mathrm{d}s,\\
\varepsilon_+ &= \int_{x}^{\infty} \sigma_{-1}\,\mathrm{d}s, & \mu_+ &= \int_{x}^{\infty} \sigma_0\,\mathrm{d}s, & \mu &= \int_{-\infty}^{\infty} \sigma_0\,\mathrm{d}s.
\end{aligned} \tag{24}$$

Thus, when y_x has compact support, $\phi\exp[\imath\lambda(x-\varepsilon_-)]$, $\bar\phi\exp[-\imath\lambda(x-\varepsilon_-)]$, $\psi\exp[-\imath\lambda(x+\varepsilon_+)]$, $\bar\psi\exp[\imath\lambda(x+\varepsilon_+)]$ and $a\exp(-\imath\lambda\varepsilon)$ are entire functions of λ. By considering the integral

$$\int_C \frac{\mathrm{d}\lambda'}{(\lambda'-\lambda)}\frac{1}{a(\lambda')\exp(-\imath\lambda'\varepsilon)}(\phi_1(\lambda'), \phi_2(\lambda'))\exp[\imath\lambda'(x-\varepsilon_-)], \tag{25}$$

where the contour (C) is defined to be the contour in the complex λ' plane, starting from $\lambda' = -\infty + \imath 0^+$, passing over all zeros of $a(\lambda')$ and ending at $\lambda' = \infty + \imath 0^+$ for λ' below C; we shall derive the Gel'fand–Levitan equation for our problem.

Using equation (9) into expression (25) yields

$$\begin{aligned}
\int_C \frac{\mathrm{d}\lambda'}{(\lambda'-\lambda)}\frac{1}{a(\lambda')\exp(-\imath\lambda'\varepsilon)}&(\phi_1(\lambda'), \phi_2(\lambda'))\exp[\imath\lambda'(x-\varepsilon_-)]\\
&= \int_C \frac{\mathrm{d}\lambda'}{(\lambda'-\lambda)}(\bar\psi_1(\lambda'), \bar\psi_2(\lambda'))\exp[\imath\lambda'(x+\varepsilon_+)]\\
&\quad + \int_C \frac{\mathrm{d}\lambda'}{(\lambda'-\lambda)}\frac{b(\lambda')}{a(\lambda')}(\psi_1(\lambda'), \psi_2(\lambda'))\exp[\imath\lambda'(x+\varepsilon_+)].
\end{aligned} \tag{26}$$

The left-hand side of equation (26) reduces to

$$\mathrm{LHS} = -\imath\pi(1, \imath\sigma_{-1}/y_x)\exp(-\mu_+), \tag{27}$$

and the right-hand side to

$$\begin{aligned}
\mathrm{RHS} = {}&-2\pi\imath(\bar\psi_1(\lambda), \bar\psi_2(\lambda))\exp[\imath\lambda(x+\varepsilon_+)] + \imath\pi(1, \imath\sigma_{-1}/y_x)\exp(-\mu_+)\\
&+ \int_C \frac{\mathrm{d}\lambda'}{(\lambda'-\lambda)}\frac{b(\lambda')}{a(\lambda')}(\psi_1(\lambda'), \psi_2(\lambda'))\exp[\imath\lambda'(x+\varepsilon_+)].
\end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned}
(\bar\psi_1(\lambda), \bar\psi_2(\lambda))\exp[\imath\lambda(x+\varepsilon_+)] = {}&(1, \imath\sigma_{-1}/y_x)\exp(-\mu_+)\\
&+ \frac{1}{2\pi\imath}\int_C \frac{\mathrm{d}\lambda'}{(\lambda'-\lambda)}\frac{b(\lambda')}{a(\lambda')}(\psi_1(\lambda'), \psi_2(\lambda'))\exp[\imath\lambda'(x+\varepsilon_+)].
\end{aligned} \tag{29}$$

Kernels K_1 and K_2 may further be defined as follows:

$$(\psi_1, \psi_2) = (0, 1)\exp[\imath\lambda(x + \varepsilon_+(x)) - \mu_+(x)] + \int_x^\infty (\lambda K_1(x, z), K_2(x, z))\exp[\imath\lambda(z + \varepsilon_+(x)) - \mu_+(x)]\,\mathrm{d}z, \tag{30}$$

with the following conditions

$$\lim_{z\to\infty} K_1(x, z) = 0, \qquad \lim_{z\to\infty} K_2(x, z) = 0. \tag{31}$$

Comparing the partial integrated equation (30) with (23), we get

$$K_1(x, x) = \sigma_{-1}/y_x = \left(1 - \epsilon\sqrt{1 + y_x^2}\right)/y_x. \tag{32}$$

We notice here that the particular case $y_x = 0$ may render $K_1(x, x)$ undetermined. In order to get the bounded value of $K_1(x, x)$, it is necessary to impose a condition that, in contrast to equation (17), $\epsilon = +1$. $K_1(x, x)$ may then be written as

$$K_1(x, x) = \left(1 - \sqrt{1 + y_x^2}\right)/y_x = -y_x/\left(1 + \sqrt{1 + y_x^2}\right), \tag{33}$$

taking 0 as a value for $y_x = 0$. This value with equation (31) corresponds to $|x| \to \infty$. It seems also worth noting that replacing $y_x = 0$ into equation (2) yields $y = 0$, which may be observed as asymptotic. Throughout this paper, we will then consider the boundary condition $y = 0$ for $|x| \to \infty$.

Now, combining equations (29) and (30), we finally obtain the coupled Gel'fand–Levitan equations for $x \geqslant w$ as follows:

$$\begin{aligned} &K_1^\star(x, w) - F(w + x) - \int_x^\infty K_2(x, z)F(w + z)\,\mathrm{d}z = 0,\\ &K_2^\star(x, w) - \int_x^\infty K_1(x, z)F''(w + z)\,\mathrm{d}z = 0, \end{aligned} \tag{34}$$

where

$$\begin{aligned} F(z) &= \frac{1}{2\pi}\int_C \frac{b(\lambda)}{a(\lambda)}\frac{1}{\lambda}\exp[\imath\lambda(z + 2\varepsilon_+(x))]\,\mathrm{d}\lambda,\\ F''(z) &= -\frac{1}{2\pi}\int_C \frac{b(\lambda)}{a(\lambda)}\lambda\exp[\imath\lambda(z + 2\varepsilon_+(x))]\,\mathrm{d}\lambda. \end{aligned} \tag{35}$$

We note here that $F''(w, z) = \partial^2 F/\partial z^2$. The time dependence of the scattering data is found from equations (4) and (6) to be

$$a(\lambda, t) = a(\lambda, 0), \qquad b(\lambda, t) = b(\lambda, 0)\exp(-\imath t/2\lambda). \tag{36}$$

The bound states are given by the zeros of $a(\lambda)$ in the upper-half plane. When all zeros of $a(\lambda)$ in the upper-half plane are simple, $F(z)$ can be expressed as

$$F(z) = \frac{1}{2\pi}\int_{-\infty}^\infty \rho(\lambda, t)\frac{1}{\lambda}\exp[\imath\lambda(z + 2\varepsilon_+)]\,\mathrm{d}\lambda + \imath\sum_{k=1}^{k=N} c_k\frac{1}{\lambda_k}\exp[\imath\lambda_k(z + 2\varepsilon_+(x))], \tag{37}$$

where

$$c_k(t) = c_k(0)\exp(-\imath t/2\lambda_k), \qquad \rho(\lambda, t) = \rho(\lambda, 0)\exp(-\imath t/2\lambda_k). \tag{38}$$

Giving the scattering data $\{\rho(\lambda, 0), \lambda; c_k(0), \lambda_k, k = 1, \ldots, N\}$, we can get $F(z)$ and then obtain $K_1(x, x)$ from the Gel'fand–Levitan equations. This yields the soliton solution by means of equation (32).

3. The N-soliton solution

We now discuss the N-soliton solution. The N-soliton solution is obtained under the conditions

$$\rho(\lambda, t) = 0, \qquad \text{and} \qquad \lambda_k^2 < 0, \tag{39}$$

so that $\lambda_k = \imath \eta_k$, where η_k is a real parameter. This gives, with equation (37),

$$F(z, t) = \sum_{k=1}^{N} \frac{c_k(t)}{\eta_k} \exp[-\eta_k(z + 2\varepsilon_+(x))], \tag{40}$$

where c_k are real.

In order to solve the Gel'fand–Levitan equation, the kernels K_1 and K_2 may take the following forms:

$$\begin{aligned} K_1(x, z) &= \sum_{k=1}^{N} A_k(x) \exp[-\eta_k(x + z + 2\varepsilon_+(x))], \\ K_2(x, z) &= \sum_{k=1}^{N} B_k(x) \exp[-\eta_k(x + z + 2\varepsilon_+(x))], \end{aligned} \tag{41}$$

where A_k and B_k are real functions.

Substituting equations (40) and (41) into (34), we get

$$\begin{aligned} A_k - \frac{c_k}{\eta_k}\left[\sum_{l=1}^{N} B_l \frac{\exp[-2\eta_l(x + \varepsilon_+(x))]}{\eta_k + \eta_l}\right] &= \frac{c_k}{\eta_k}, \\ B_k - \eta_k c_k \left[\sum_{l=1}^{N} A_l \frac{\exp[-2\eta_l(x + \varepsilon_+(x))]}{\eta_k + \eta_l}\right] &= 0. \end{aligned} \tag{42}$$

The A_k are obtained as

$$A_k = \frac{D_k}{D}, \tag{43}$$

where D represents the determinant of the cœ fficient matrix defined as

$$D = \begin{vmatrix} I & F \\ G & I \end{vmatrix}, \tag{44}$$

with

$$\begin{aligned} F_{kl} &= -\frac{c_k}{\eta_k(\eta_k + \eta_l)} \exp[-2\eta_l(x + \varepsilon_+(x))], \qquad I_{kl} = \delta_{kl}, \\ G_{kl} &= -\frac{\eta_k c_k}{(\eta_k + \eta_l)} \exp[-2\eta_l(x + \varepsilon_+(x))], \end{aligned} \tag{45}$$

while the determinant D_k is defined as

$$\begin{vmatrix} 1 & 0 & . & 0 & c_1/\eta_1 & 0 & . & . & . & . & F_{11} & F_{12} & \cdots & F_{1N} \\ . & . & . & . & . & . & . & . & . & . & F_{21} & F_{22} & \cdots & F_{2N} \\ . & . & . & 1 & c_{k-1}/\eta_{k-1} & 0 & . & . & . & . & F_{k-11} & F_{k-12} & \cdots & F_{k-1N} \\ . & . & . & 0 & c_k/\eta_k & 0 & . & . & . & . & F_{k1} & F_{k2} & \cdots & F_{kN} \\ . & . & . & 0 & c_{k+1}/\eta_{k+1} & 1 & . & . & . & . & F_{k+11} & F_{k+12} & \cdots & F_{k+1N} \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 1 & F_{N1} & F_{N2} & \cdots & F_{NN} \\ G_{11} & G_{12} & . & . & G_{1k-1} & 0 & G_{1k+1} & . & . & G_{1N} & 1 & . & . & 0 \\ G_{21} & G_{22} & . & . & G_{2k-1} & 0 & G_{2k+1} & . & . & G_{2N} & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 0 & . & . & . & 0 \\ G_{N1} & G_{N2} & . & . & G_{Nk-1} & 0 & G_{Nk+1} & . & . & G_{NN} & 0 & . & . & 1 \end{vmatrix}. \tag{46}$$

Thus,

$$K_1(x,x) = \sum_{k=1}^{N} A_k(x) \exp[-2\eta_k(x+\varepsilon_+(x))]. \tag{47}$$

Using equation (32), we get

$$y_x = 2\frac{K_1(x,x)}{K_1^2(x,x)-1}. \tag{48}$$

Making use of equations (20) and (24), we find that

$$[x+\varepsilon_+(x)]_x = \sqrt{1+y_x^2}. \tag{49}$$

Introducing some variable s defined as

$$s_x = \sqrt{1+y_x^2}, \tag{50}$$

equation (49) is written in the form

$$[x+\varepsilon_+(x)]_x = s_x, \tag{51}$$

so that the meaning of $[x+\varepsilon_+(x)]$ then becomes clear. Therefore, equation (49) is expressed as

$$[x+\varepsilon_+(x)]_x = \frac{1+K_1^2(x,x)}{1-K_1^2(x,x)}. \tag{52}$$

As both $K_1(x,x)$ and y_x are functions of $[x+\varepsilon_+(x)]$, it is better to transform from the independent variable x to the arc length

$$u = x + \varepsilon_+(x), \tag{53}$$

when investigating the soliton solution. Using the independent variable u, we obtain

$$y_u = -\frac{2K_1(u)}{1+K_1^2(u)} \tag{54}$$

and

$$[\varepsilon_+(u)]_u = \frac{2K_1^2(u)}{1+K_1^2(u)}. \tag{55}$$

On account of this parametrization of the position along the solution curve, it is possible to express the soliton solutions as a single-valued function if the solitons are described in terms of u.

Throughout this paper, we pay some attention to finite solutions with the following boundary conditions:

$$\left.\begin{array}{l} y(u) \to 0 \\ \varepsilon_+(u) \to 0 \end{array}\right\} \qquad \text{as} \quad u \to \infty. \tag{56}$$

Integrating equations (54) and (55) with respect to u, we finally arrive at the N-soliton solution

$$y = -\int_{-\infty}^{u} \mathrm{d}u \frac{2K_1(u)}{1+K_1^2(u)}, \tag{57}$$

and

$$\varepsilon_+(u) = \int_{-\infty}^{u} \mathrm{d}u \frac{2K_1^2(u)}{1+K_1^2(u)}, \tag{58}$$

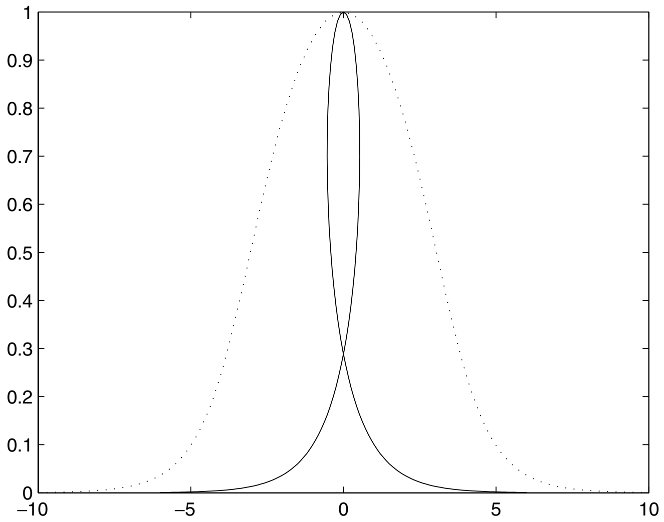

Figure 1. Loop (solid line) and hump (dotted line) solitons.

where $K_1(u)$ is one of the solutions of the Gel'fand–Levitan equations, and is given by

$$K_1(u) = \sum_{k=1}^{N} A_k(u)\exp(-2\eta_k u), \tag{59}$$

and

$$A_k - \frac{c_k}{\eta_k}\left[\sum_{l=1}^{N}\sum_{m=1}^{N} c_l\eta_l A_m \frac{\exp[-2(\eta_l+\eta_m)u]}{(\eta_k+\eta_l)(\eta_l+\eta_m)}\right] = \frac{c_k}{\eta_k}. \tag{60}$$

The last two equations are derived from equations (42) and (47). We note that the parameters $\eta_k (k = 1, \ldots, N)$ merely denote the eigenvalues of the soliton solutions.

We pay some attention to the special cases $N = 1$ and $N = 2$, which refer to one- and two-soliton solutions, respectively. Their expressions may be easily derived from equations (57) and (58) after small calculations. As a result, the one-soliton solutions ηy versus $2\eta\xi$ with $\xi = x - vt$, v being the velocity, depicted in figure 1, are of two types, loop- and hump-like represented by solid and dotted lines, respectively. These two structures may be related by the transformation $\eta \to -\eta$, provided that the boundary condition given by equation (56) is satisfied. Indeed, with the relation $\lambda = -\imath\eta(\eta > 0)$ introduced in the WKI-method while investigating the one-soliton solutions, the hump-like solution is found whereas the other solution of loop-like shape, is related to $\lambda = \imath\eta(\eta > 0)$. The maximum amplitude and phase velocity in the (u, t)-spacetime of the previous structures are given by $1/\eta$ and $-1/(2\eta^2)$, respectively. Thus, a larger loop or hump soliton always moves faster than a smaller one in the negative direction of (u, t)-spacetime. It seems worth noting that the soliton solution with negative amplitude may be derived from the positive one simply by changing y into $-y$. The soliton solution with negative amplitude is conventionally called anti-soliton solution.

Concerning the two-soliton solutions, we study the behaviour of $y(x, t)$ versus x at some given time t. We find two kinds of features depending on the ratio of the two eigenvalues

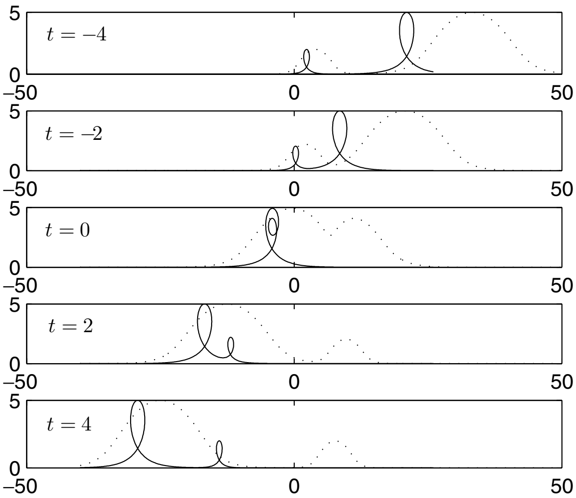

Figure 2. The collision process for two loop-like (solid line) and two hump-like (dotted line) solitons with $\eta_1 = 0.5$ and $\eta_2 = 0.2$.

η_1 and η_2 characterizing each single soliton. These features which are due to the nonlinear interaction may be described by the shifts Δx_1 and Δx_2 of the individual solitons. These shifts however satisfy the following relation:

$$\sum_{j=1}^{2} \eta_j \Delta x_j \neq 0, \tag{61}$$

which shows that the centre of mass is not a constant of motion.

We illustrate three representative cases as follows:

- two-soliton solutions with 'dissimilar' amplitudes as shown in figure 2;
- soliton and anti-soliton solutions as shown in figure 3.
- two-soliton solutions with 'similar' amplitudes as shown in figure 4;

In these figures, loop and hump soliton solutions are represented by solid and dotted lines, respectively.

In figure 2, a larger soliton for $\eta_2 = 0.2$ with the faster velocity and larger energy attracts the smaller one for $\eta_1 = 0.5$ which travels along before being shifted after the interaction. This kind of scattering feature may also be seen in figure 3 where the small soliton $\eta_1 = 0.3$ travels along the anti-soliton $\eta_2 = -0.2$.

In figure 4, the solitons attract and collide. The larger and faster soliton for $\eta_2 = 0.2$ with the larger energy pushes by contact the smaller one for $\eta_1 = 0.3$ in such a way that the solitons do not overlap, though they seem to exchange their amplitudes during the period of the nonlinear interaction.

We have further plotted many curves $y(x, t)$ versus x corresponding to different values of the ratio $\eta_2/\eta_1 (0 < \eta_2 < \eta_1)$, but without any report in the present paper for some convenience. As a result, we find that soliton solutions with 'dissimilar' amplitudes ($\eta_2/\eta_1 < 0.6$) attract elastically in a way that the smaller soliton always travels along the larger one. Solitons with 'similar' amplitudes ($0.6 \leqslant \eta_2/\eta_1 \leqslant 1$) seem to repel and exchange their amplitudes during the scattering process. We should also note that in the case of $\eta_2 < 0 < \eta_1$, the smaller soliton always travels around the larger one even if their magnitudes are 'similar'.

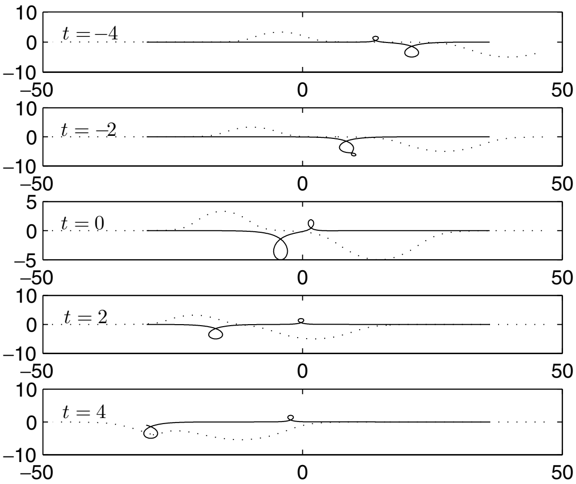

Figure 3. The collision process of a soliton for $\eta_1 = 0.3$ and an anti-soliton for $\eta_2 = -0.2$. Loop and hump are represented by solid and broken lines, respectively.

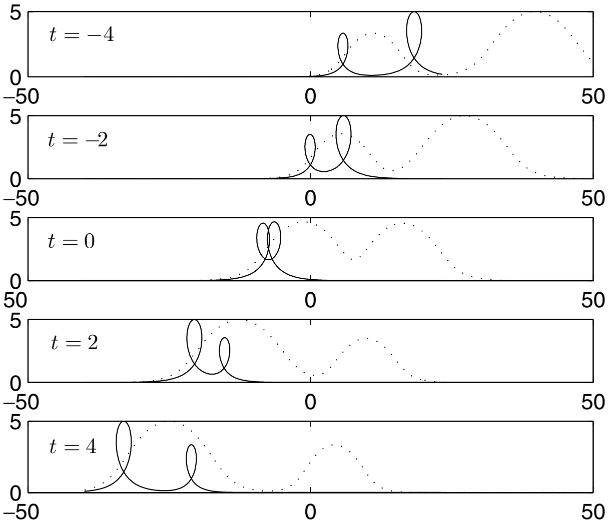

Figure 4. The collision process for two loop-like (solid line) and two hump-like (dotted line) solitons with $\eta_1 = 0.3$ and $\eta_2 = 0.2$.

4. Summary

We have investigated a partial differential equation of a new type recently derived by Schäfer and Wayne [15] studying the propagation of an ultrashort pulse in a nonlinear medium. We have studied the propagation of loop and hump soliton solutions characterized by the previous novel equation. The corresponding *N*-soliton solutions have been found analytically by means of an inverse scattering method. Performing a more detailed analysis, the properties of the one- and two-soliton solutions have been studied. This method may arguably be more efficient and straightforward than the map through the sine-Gordon equation investigated much earlier by Sakovich and Sakovich [16]. As a result, we have found that when two-soliton solutions of the previous equation with 'similar' or 'dissimilar' amplitudes collide, they always shift backwards except when one of them has a negative amplitude. We have observed that, with these special soliton solutions, the interaction type depends on the ratio of the two eigenvalues involved. We have also shown that the two basic collision processes with 'similar' and 'dissimilar' amplitudes, respectively, depend strongly upon a critical value of the ratio of the two eigenvalues.

Moreover, for some simplicity in our analysis, we have considered, from equation (39), that $\rho(\lambda, t) = 0$ when discussing the N-soliton solution. We can also investigate in detail the case corresponding to $\rho(\lambda, t) \neq 0$. This may lead us to a kind of soliton which seems to breathe. This kind of solution may be composed of an envelop with hump-like shape and a carrier with oscillating shape. This may constitute another interesting investigation which will certainly lead us to an essential criterion as we have previously found in this paper.

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